MATHEMATICAL METHODS (CAS)

Written examination 2

Monday 12 November 2007

Reading time: 11.45 am to 12.00 noon (15 minutes)
Writing time: 12.00 noon to 2.00 pm (2 hours)

QUESTION AND ANSWER BOOK

Structure of book

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<th>Number of marks</th>
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<tr>
<td>1</td>
<td>22</td>
<td>22</td>
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<td>2</td>
<td>5</td>
<td>5</td>
<td>58</td>
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<td>Total 80</td>
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</tbody>
</table>

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved CAS calculator (memory DOES NOT need to be cleared) and, if desired, one scientific calculator. For approved computer based CAS, their full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied
- Question and answer book of 23 pages with a detachable sheet of miscellaneous formulas in the centrefold.
- Answer sheet for multiple-choice questions.

Instructions
- Detach the formula sheet from the centre of this book during reading time.
- Write your student number in the space provided above on this page.
- Check that your name and student number as printed on your answer sheet for multiple-choice questions are correct, and sign your name in the space provided to verify this.
- All written responses must be in English.

At the end of the examination
- Place the answer sheet for multiple-choice questions inside the front cover of this book.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

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SECTION 1

Instructions for Section 1
Answer all questions in pencil on the answer sheet provided for multiple-choice questions. Choose the response that is correct for the question. A correct answer scores 1, an incorrect answer scores 0. Marks will not be deducted for incorrect answers. No marks will be given if more than one answer is completed for any question.

Question 1
The linear function \( f : D \rightarrow R \), \( f(x) = 6 - 2x \) has range \([-4, 12]\). The domain \( D \) is
A. \([-3, 5]\)
B. \([-5, 3]\)
C. \(R\)
D. \([-14, 18]\)
E. \([-18, 14]\)

Question 2
Let \( g(x) = x^2 + 2x - 3 \) and \( f(x) = e^{2x + 3} \). Then \( f(g(x)) \) is given by
A. \( e^{4x^2 + 6} + 2e^{2x^3} - 3 \)
B. \( 2x^2 + 4x - 6 \)
C. \( e^{2x^2 + 4x + 9} \)
D. \( e^{2x^2 + 4x} - 3 \)
E. \( e^{2x^2 + 4x} - 6 \)

Question 3
If \( y = \log_a(7x - b) + 3 \), then \( x \) is equal to
A. \( \frac{1}{7} a^{y-3} + b \)
B. \( \frac{1}{7} (a^y - 3) + b \)
C. \( \frac{1}{7} (a^y - 3 + b) \)
D. \( a^{y-3} - \frac{b}{7} \)
E. \( \frac{y - 3}{\log_a(7-b)} \)
**Question 4**
The average rate of change of the function with rule \( f(x) = x^3 - \sqrt{x + 1} \) between \( x = 0 \) and \( x = 3 \) is
A. 0
B. 12
C. \( \frac{26}{3} \)
D. \( \frac{25}{3} \)
E. 8

**Question 5**
The simultaneous linear equations
\[
mx + 12y = 24 \\
3x + my = m
\]
have a unique solution only for
A. \( m = 6 \) or \( m = -6 \)
B. \( m = 12 \) or \( m = 3 \)
C. \( m \in \mathbb{R} \setminus \{-6, 6\} \)
D. \( m = 2 \) or \( m = 1 \)
E. \( m \in \mathbb{R} \setminus \{-12, -3\} \)

**Question 6**
The range of the function \( f : \left[ 0, \frac{\pi}{3} \right) \rightarrow \mathbb{R} \), \( f(x) = 3 \left| \sin (2x) - 1 \right| + 2 \) is
A. \([2, 5)\)
B. \([2, 5]\)
C. \( \left[ 5 - \frac{3\sqrt{3}}{2}, 5 \right) \)
D. \( \left( 5 - \frac{3\sqrt{3}}{2}, 5 \right] \)
E. \((2.5, 5]\)
Question 7
The random variable $X$ has a normal distribution with mean 11 and standard deviation 0.25.
If the random variable $Z$ has the standard normal distribution, then the probability that $X$ is less than 10.5 is equal to
A. $\Pr(Z > 2)$
B. $\Pr(Z < -1.5)$
C. $\Pr(Z < 1)$
D. $\Pr(Z \geq 1.5)$
E. $\Pr(Z < -4)$

Question 8
Which one of the following is not true about the function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = |2x + 4|$?
A. The graph of $f'$ is continuous everywhere.
B. The graph of $f'$ is continuous everywhere.
C. $f(x) \geq 0$ for all values of $x$
D. $f'(x) = 2$ for all $x > 0$
E. $f'(x) = -2$ for all $x < -2$

Question 9
Let $k = \int_{-2}^{1} \frac{1}{x} \, dx$, then $e^k$ is equal to
A. $\log_e(2)$
B. 1
C. 2
D. $e$
E. $\frac{1}{2}$

Question 10
The graph of $y = kx - 3$ intersects the graph of $y = x^2 + 8x$ at two distinct points for
A. $k = 11$
B. $k > 8 + 2\sqrt{3}$ or $k < 8 - 2\sqrt{3}$
C. $5 \leq k \leq 6$
D. $8 - 2\sqrt{3} \leq k \leq 8 + 2\sqrt{3}$
E. $k = 5$

Question 11
The solution set of the equation $e^{4x} - 5e^{2x} + 4 = 0$ over $\mathbb{R}$ is
A. $\{1, 4\}$
B. $\{-4, -1\}$
C. $\{-2, -1, 1, 2\}$
D. $\{-\log_e(2), 0, \log_e(2)\}$
E. $\{0, \log_e(2)\}$
Question 12
Let \( f: \mathbb{R} \rightarrow \mathbb{R} \) be a differentiable function such that
\[
\begin{align*}
& f'(3) = 0 \\
& f'(x) < 0 \text{ when } x < 3 \text{ and when } x > 3
\end{align*}
\]
When \( x = 3 \), the graph of \( f \) has a
A. local minimum.
B. local maximum.
C. stationary point of inflection.
D. point of discontinuity.
E. gradient of 3.

Question 13
For the graph of \( y = 4x^3 + 27x^2 - 30x + 10 \) the subset of \( \mathbb{R} \) for which the gradient is negative is given by the interval
A. \((0.5, 5.0)\)
B. \((-4.99, 0.51)\)
C. \((-\infty, \frac{1}{2})\)
D. \((-5, \frac{1}{2})\)
E. \((2.25, \infty)\)

Question 14
The maximal domain \( D \) of the function \( f: D \rightarrow \mathbb{R} \) with rule
\[
f(x) = \log_e(|x - 3|) + 6
\]
is
A. \( \mathbb{R} \setminus \{3\} \)
B. \((3, \infty)\)
C. \( \mathbb{R} \)
D. \((-3, \infty)\)
E. \((-\infty, 3)\)
Question 15
The graph of the function \( f: [0, \infty) \to \mathbb{R} \) where \( f(x) = 3x^\frac{5}{2} \) is reflected in the \( x \)-axis and then translated 3 units to the right and 4 units down.
The equation of the new graph is
A. \( y = 3(x - 3)^{\frac{5}{2}} + 4 \)
B. \( y = -3(x - 3)^{\frac{5}{2}} - 4 \)
C. \( y = -3(x + 3)^{\frac{5}{2}} - 1 \)
D. \( y = -3(x - 4)^{\frac{5}{2}} + 3 \)
E. \( y = 3(x - 4)^{\frac{5}{2}} + 3 \)

Question 16
If a random variable \( X \) has probability density function
\[
f(x) = \begin{cases} \frac{x}{2} & \text{if } x \in [0, 2] \\ 0 & \text{otherwise} \end{cases}
\]
then \( E(X) \) is equal to
A. \( \frac{1}{2} \)
B. 1
C. \( \frac{4}{3} \)
D. \( \frac{2}{3} \)
E. 2

Question 17
The function \( f \) satisfies the functional equation \( f(f(x)) = x \) for the maximal domain of \( f \).
The rule for the function is
A. \( f(x) = x + 1 \)
B. \( f(x) = x - 1 \)
C. \( f(x) = \frac{x - 1}{x + 1} \)
D. \( f(x) = \log_e(x) \)
E. \( f(x) = \frac{x + 1}{x - 1} \)
Question 18
The heights of the children in a queue for an amusement park ride are normally distributed with mean 130 cm and standard deviation 2.7 cm. 35% of the children are not allowed to go on the ride because they are too short. The minimum acceptable height correct to the nearest centimetre is
A. 126
B. 127
C. 128
D. 129
E. 130

Question 19
The discrete random variable $X$ has probability distribution as given in the table. The mean of $X$ is 5.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pr($X = x$)</td>
<td>$a$</td>
<td>0.2</td>
<td>0.2</td>
<td>0.3</td>
<td>$b$</td>
</tr>
</tbody>
</table>

The values of $a$ and $b$ are
A. $a = 0.05$ and $b = 0.25$
B. $a = 0.1$ and $b = 0.29$
C. $a = 0.2$ and $b = 0.9$
D. $a = 0.3$ and $b = 0$
E. $a = 0$ and $b = 0.3$

Question 20
The average value of the function $y = \tan(2x)$ over the interval $[0, \frac{\pi}{8}]$ is
A. $\frac{2}{\pi} \log_e (2)$
B. $\frac{\pi}{4}$
C. $\frac{1}{2}$
D. $\frac{4}{\pi} \log_e 2$
E. $\frac{8}{\pi}$

Question 21
$\{x: \cos^2(x) + 2\cos(x) = 0\} =$
A. $\{x : \cos(x) = 0\}$
B. $\{x : \cos(x) = -\frac{1}{2}\}$
C. $\{x : \cos(x) = \frac{1}{2}\}$
D. $\{x : \cos(x) = 0\} \cup \{x : \cos(x) = -\frac{1}{2}\}$
E. $\{x : \cos(x) = \frac{1}{2}\} \cup \{x : \cos(x) = -\frac{1}{2}\}$
Question 22
The graphs of $y = f(x)$ and $y = g(x)$ are as shown.

The graph of $y = f(g(x))$ is best represented by

A. 

B. 

C. 

D. 

E.
Working space
Question 1
A dog food company manufactures dog food in cylindrical cans of volume $V$ cm$^3$.
The height of the can is $h$ cm and the radius of the can is $r$ cm, where $\frac{\sqrt{10}}{9} \leq r \leq \frac{5\sqrt{1}}{9}$.

a. Express $h$ in terms of $r$ and $V$.

b. Show that the total surface area, $A$ cm$^2$, of the can is given by $A = \frac{2V}{r} + 2\pi r^2$.

c. Find, in terms of $V$, the value of $r$ so that the surface area of the can is a minimum. (You are not required to justify the nature of the stationary point.)
d. Find, in terms of $V$, the maximum possible surface area of the can.

2 marks
Total 7 marks
Question 2

Tasmania Jones is attempting to recover the lost Zambeji diamond. The diamond is buried at a point 4 km into Death Gorge, which is infested with savage insects. In order to recover the diamond, Tasmania will need to run into the gorge, dig up the diamond and return the same way that he came.

The concentration of insects in the gorge is a continuous function of time. The concentration $C$, insects per square metre, is given by

$$C(t) = \begin{cases} 
1000\cos\left(\frac{\pi(t-8)}{2}\right) + 2)^2 - 1000 & 8 \leq t \leq 16 \\
\frac{m}{t} & 0 \leq t < 8 \text{ or } 16 < t \leq 24 
\end{cases}$$

where $t$ is the number of hours after midnight and $m$ is a real constant.

a. What is the value of $m$?

b. Sketch the graph of $C$ for $0 \leq t \leq 24$
c. What is the minimum concentration of insects and at what value(s) of \( t \) does that occur?

The insects infesting the gorge are known to be deadly if their concentration is more than 1250 insects per square metre.

d. At what time after midnight does the concentration of insects first stop being deadly?

e. During a 24-hour period, what is the total length of time for which the concentration of insects is less than 1250 insects per square metre?
Due to the uneven surface of the gorge, the time, $T$ minutes, that Tasmania will take to run $x$ km into the gorge is given by $T = p(q^x - 1)$, where $p$ and $q$ are constants.

f. Tasmania knows that it will take him 5 minutes to run the first kilometre and 12.5 minutes to run the first two kilometres.

i. Find the values of $p$ and $q$.

ii. Find the length of time that Tasmania will take to run the 4 km to reach the buried diamond.

2 + 1 = 3 marks

g. Tasmania takes 19 minutes to dig up the diamond and he is able to run back through the gorge in half the time it took him to reach the diamond. Show that it is possible for him to recover the diamond successfully and state how much time he has to spare.

3 marks

Total 15 marks
Working space
Question 3

Shown below are the graphs of the functions

\[ f: [0, 4] \rightarrow \mathbb{R}, \quad f(x) = 6 \sin \left( \frac{\pi x}{2} \right) \]

\[ g: [0, 4] \rightarrow \mathbb{R}, \quad g(x) = 2x(x - 2)(x - 4) = 2(x^3 - 6x^2 + 8) \]

The point (1, 6) lies on both graphs.

a. Find the exact value of \( x \) for which \( g(x) \) is maximum.

\[ \text{Answer: } x = 3 \]

2 marks
b. i. Write an expression for the total area of the shaded regions using definite integrals.

ii. Find the total area of these shaded regions.

2 + 2 = 4 marks

c. Find, correct to two decimal places, the maximum value of $|f(x) - g(x)|$ for $0 \leq x \leq 4$ and the values of $x$ in this interval for which this maximum occurs.

2 marks

d. Let $h: [-3, 1] \to \mathbb{R}, h(x) = 6 \cos\left(\frac{\pi x}{2}\right)$.
   i. State a sequence of two transformations which takes the graph of $f$ to the graph of $h$. 
ii. Hence or otherwise, find a cubic polynomial function with domain \([-3, 1]\) that has the same \(x\)-intercepts as \(h\) and the same maximum and minimum values as \(g\).

\[
\text{2 + 2 = 4 marks}
\]

Total 12 marks
Working space
Question 4
Part of the graph of the function $f: R \rightarrow R$, $f(x) = a + be^{-x}$ is shown below. The line with equation $y = 1$ is an asymptote and the graph passes through the origin.

![Graph of function $f$ and line $y = 1$.](image)

a. Explain why $a = 1$ and $b = -1$.

\[
\begin{align*}
\text{Let } h \text{ be the function } h: [0, 2] \rightarrow R, h(x) = f(x). \\
\text{b. State the range of } h \text{ using exact values.}
\end{align*}
\]

2 marks
c. i. Find the inverse function \( h^{-1} \).

\[
\text{\textbf{\textit{Answer here.}}} 
\]

ii. Sketch and label the graph of the inverse function \( h^{-1} \) on the axes on page 20.

\[
1 + 2 = 3 \text{ marks} 
\]

d. For \( f(x) = 1 - e^{-x} \)

i. show that \( f(u)f(v) = f(u) + f(v) - f(u + v) \), where \( u \) and \( v \) are real numbers

\[
\text{\textbf{\textit{Answer here.}}} 
\]

ii. show that \( f(u)f(-u) = f(u) + f(-u) \), where \( u \) is a real number.

\[
\text{\textbf{\textit{Answer here.}}} 
\]

\[
2 + 1 = 3 \text{ marks} 
\]

Total 10 marks
Question 5
In the Great Fun amusement park there is a small train called Puffing Berty which does a circuit of the park. The continuous random variable $T$, the time in minutes for a circuit to be completed, has a probability density function $f$ with rule

$$f(t) = \begin{cases} 
\frac{1}{100}(t-10) & \text{if } 10 \leq t < 20 \\
\frac{1}{100}(30-t) & \text{if } 20 \leq t \leq 30 \\
0 & \text{otherwise}
\end{cases}$$

a. Sketch the graph of $y = f(t)$ on the axes provided.

b. Find the probability that the time taken by Puffing Berty to complete a full circuit is less than 25 minutes. (Give the exact value.)

c. Find $\Pr(T \leq 15 \mid T \leq 25)$. (Give the exact value.)
The train must complete six circuits between 9.00 am and noon. The management prefers Puffing Berty to complete a circuit in less than 25 minutes.

d. Find the probability, correct to four decimal places, that of the 6 circuits completed, at least 4 of them take less than 25 minutes each.

For scheduling reasons the management wants to know the time, $b$ minutes, for which the probability of exactly 3 or 4 out of the 6 circuits completed each taking less than $b$ minutes, is maximised.

Let $Pr(T < b) = p$
Let $Q$ be the probability that exactly 3 or 4 circuits completed each take less than $b$ minutes.

e. Show that $Q = 5p^3(1 - p)^2 (4 - p)$.

f. i. Find the maximum value of $Q$ and the value of $p$ for which this occurs. (Give the exact value.)

ii. Find, correct to one decimal place, the value of $b$ for which this maximum occurs.

Total 14 marks
MATHEMATICAL METHODS AND MATHEMATICAL METHODS (CAS)

Written examinations 1 and 2

FORMULA SHEET

Directions to students

Detach this formula sheet during reading time.
This formula sheet is provided for your reference.
This page is blank
Mathematical Methods and Mathematical Methods CAS
Formulas

Mensuration

area of a trapezium: \( \frac{1}{2} (a + b)h \)

volume of a pyramid: \( \frac{1}{3} Ah \)

curved surface area of a cylinder: \( 2\pi rh \)

volume of a cylinder: \( \pi r^2 h \)

area of a triangle: \( \frac{1}{2} bc \sin A \)

Calculus

\[
\frac{d}{dx} \left( x^n \right) = nx^{n-1}
\]

\[
\frac{d}{dx} \left( e^{ax} \right) = ae^{ax}
\]

\[
\frac{d}{dx} \left( \log_e(x) \right) = \frac{1}{x}
\]

\[
\frac{d}{dx} \left( \sin(ax) \right) = a \cos(ax)
\]

\[
\frac{d}{dx} \left( \cos(ax) \right) = -a \sin(ax)
\]

\[
\frac{d}{dx} \left( \tan(ax) \right) = \frac{a}{\cos^2(ax)} = a \sec^2(ax)
\]

product rule: \( \frac{d}{dx} (uv) = u \frac{dv}{dx} + v \frac{du}{dx} \)

quotient rule: \( \frac{d}{dx} \left( \frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \)

chain rule: \( \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \)

approximation: \( f(x + h) = f(x) + hf'(x) \)

Probability

Pr\( (A) = 1 - \Pr(A') \)

Pr\( (A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B) \)

Pr\( (A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} \)

mean: \( \mu = \mathbb{E}(X) \)

variance: \( \text{var}(X) = \sigma^2 = \mathbb{E}((X - \mu)^2) = \mathbb{E}(X^2) - \mu^2 \)

<table>
<thead>
<tr>
<th>probability distribution</th>
<th>mean</th>
<th>variance</th>
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</thead>
<tbody>
<tr>
<td>discrete ( \Pr(X = x) = p(x) )</td>
<td>( \mu = \sum x , p(x) )</td>
<td>( \sigma^2 = \sum (x - \mu)^2 , p(x) )</td>
</tr>
<tr>
<td>continuous ( \Pr(a &lt; X &lt; b) = \int_a^b f(x) , dx )</td>
<td>( \mu = \int_{-\infty}^{\infty} x , f(x) , dx )</td>
<td>( \sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 , f(x) , dx )</td>
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</tbody>
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END OF FORMULA SHEET